USAGE OF *OKUBO*

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Contents

1. Introduction

The program `Okubo` has been developing by the author to analyze the combinatorial structure of local systems attached to tuples of partitions.

In particular Conjecture Rigid 3.1.1 for rigid local systems can be checked by this program. Moreover the reductions by Katz, Yokoyama and the author can be shown by the program.

If you find any problem or a bug in this program, please inform it to the author.

2. Rigidity

Let $\mathbf{m} = (m_{0,1}, \ldots, m_{0,n_0}; \ldots; m_{k,1}, \ldots, m_{k,n_k})$ be a $(k+1)$-tuple of partitions of a positive integer n:

$$n = m_{j,1} + \cdots + m_{j,n_j} \quad (j = 0, \ldots, k)$$

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and let $\mathcal{P}_{k+1}^{(n)}$ be the totality of such $(k+1)$-tuples. Put

$$\begin{aligned}
&\mathcal{P}_{k+1} = \bigcup_{n=0}^{\infty} \mathcal{P}_{k+1}^{(n)}, \quad \mathcal{P}^{(n)} = \bigcup_{k=2}^{\infty} \mathcal{P}_{k+1}^{(n)}, \\
&\operatorname{ord} \mathbf{m} = n, \\
&\lambda_{\mathbf{m}} = \big(\lambda_{j,\nu}\big)_{\substack{0\le j\le k\\ 1\le \nu\le n_j}} \in \mathbb{C}^{n_0+\cdots+n_k}
\end{aligned}$$

and

$$\begin{aligned}
V_{\mathbf{m},\lambda_{\mathbf{m}}} := \Big\{(A_1,\ldots,A_k) \in M(n,\mathbb{C})^k \,;\, A_j \sim \bigoplus_{\nu=1}^{n_j} \lambda_{j,\nu} I_{m_{j,\nu}} \quad (1 \le j \le n_j)\\
-(A_1+\cdots+A_k) \sim \bigoplus_{\nu=1}^{n_0} \lambda_{0,\nu} I_{m_{0,\nu}}\Big\}
\end{aligned}$$

with the $GL(n,\mathbb{C})$-action:

$$\begin{array}{ccc}
GL(n,\mathbb{C}) \times V_{\mathbf{m},\lambda_{\mathbf{m}}} & \to & V_{\mathbf{m},\lambda_{\mathbf{m}}} \\
\cup & & \cup \\
\big(g,(A_1,\ldots,A_k)\big) & \mapsto & \big(gA_1g^{-1},\ldots,gA_kg^{-1}\big).
\end{array}$$

Denoting $A_0 = -(A_1+\cdots+A_k)$, we have

$$\sum_{j=0}^{k}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu} = 0 \tag{2.1}$$

if $V_{\mathbf{m},\lambda_{\mathbf{m}}} \neq \emptyset$.

Definition 2.1. i) $(A_1,\ldots,A_k) \in M(n,\mathbb{C})^k$ is *irreducible* if

$$A_jV \subset V \quad (j=1,\ldots,k) \text{ with a subspace } V \text{ of } \mathbb{C}^n \ \Rightarrow\ \dim V = 0 \text{ or } n.$$

ii) A tuple $\mathbf{m}$ is *irreducible* if $m_{j,\nu}$ have no non-trivial common divisor ($1 \le \nu \le n_j$, $0 \le j \le k$).

iii) A tuple $\mathbf{m}$ and $V_{\mathbf{m},\lambda_{\mathbf{m}}}$ are called *realizable* if $V_{\mathbf{m},\lambda_{\mathbf{m}}}$ is non-empty for a generic $\lambda_{\mathbf{m}}$ satisfying (2.1).

iv) $V_{\mathbf{m},\lambda_{\mathbf{m}}}$ is called *rigid* if it is a single $GL(n,\mathbb{C})$-orbit and $\mathbf{m}$ is called *rigid* if $V_{\mathbf{m},\lambda_{\mathbf{m}}}$ is non-empty and rigid for a generic $\lambda_{\mathbf{m}}$ satisfying (2.1).

Theorem 2.2 (Katz)**.** i) *Suppose* $\mathbf{A} = (A_1,\ldots,A_k) \in M(n,\mathbb{C})^k$ *is irreducible. Put* $A_0 = -(A_1+\cdots+A_k)$. *Then*

$$\begin{aligned}
V_{\mathbf{A}} := \Big\{(B_1,\ldots,B_k) \in M(n,\mathbb{C})^k \,;\, B_j \sim A_j \ (j=1,\ldots,k)\\
\textit{and} \ \sum_{j=1}^{k} B_j \sim \sum_{j=1}^{k} A_j\Big\}
\end{aligned}$$

is a single $GL(n,\mathbb{C})$*-orbit if and only if the rigidity index*

$$\operatorname{Ridx}(\mathbf{A}) := \sum_{j=0}^{k} \dim\{A \in M(n,\mathbb{C})\,;\, AA_j = A_jA\} - (k-1)n^2 \tag{2.2}$$

equals 2.

ii) *Let* $\mathbf{m}$ *be a* $(k+1)$*-tuple of partitions of* n *with* $n>1$. *We may assume*

$$m_{j,1} \ge m_{j,2} \ge \cdots \ge m_{j,n_j} > 0 \quad (j=0,\ldots,k) \tag{2.3}$$

by arranging $\mathbf{m}$ *if necessary. Then* $\mathbf{m}$ *is rigid if and only if*

$$(m_{0,1}+\cdots+m_{k,1})-m_{j,1}\le \operatorname{ord}\mathbf{m}< m_{0,1}+\cdots+m_{k,1}\quad (j=0,\ldots,k) \tag{2.4}$$

and moreover $\mathcal{K}\mathbf{m}$ *is rigid.*

Here $\mathcal{K}\mathbf{m}$ *is obtained by replacing* $m_{j,1}$ *by* $m_{j,1}-d$ *with* $d:=m_{0,1}+\cdots+m_{k,1}-\operatorname{ord}\mathbf{m}$ *for* $j=0,\ldots,k$*. If* 0 *appears in* $\mathcal{K}\mathbf{m}$*, it may be deleted and the trivial partition appears in* $\mathcal{K}\mathbf{m}$*, it may be also deleted.*

This $\mathcal{K}$ corresponds to Katz' *middle convolution*.

Theorem 2.3. *A tuple* $\mathbf{m}\in\mathcal{P}_{k+1}$ *with* (2.3) *is realizable if and only if*

$$m_{0,1}+m_{1,1}+\cdots+m_{k,1}\le \operatorname{ord}\mathbf{m} \tag{2.5}$$

or $\mathcal{K}\mathbf{m}$ *is well-defined and* $\mathcal{K}\mathbf{m}$ *is realizable.*

Since $\operatorname{ord}\mathcal{K}\mathbf{m}=\operatorname{ord}\mathbf{m}-d$ and $\mathbf{m}$ is rigid and realizable if $\operatorname{ord}\mathbf{m}\le 1$, we have an algorithm checking the rigidity and realizability of $\mathbf{m}$.

For example

$$\begin{aligned}
&\mathcal{R}_4^{(10)}\ni 721,631,7111,55 \overset{25-2*10=5}{\longrightarrow} 221,311,2111 \overset{7-5=2}{\longrightarrow} 21,111,111\\
&\qquad\overset{4-3=1}{\longrightarrow} 11,11,11 \overset{3-2=1}{\longrightarrow} \emptyset\\
&\mathcal{R}_3^{(4)}\ni 22,22,1111 \overset{5-4=1}{\longrightarrow} 21,21,111 \overset{5-3=2}{\longrightarrow} \times \qquad\qquad \text{(not realizable)}\\
&\mathcal{R}_3^{(4)}\ni 211,211,1111 \overset{5-4=1}{\longrightarrow} 111,111,111 \overset{3-3=0}{\longrightarrow} \times \qquad \text{(realizable and non-rigid)}
\end{aligned}$$

The totality of the rigid tuples in $\mathcal{P}_{k+1}^{(n)}$ is denoted by $\mathcal{R}_{k+1}^{(n)}$ and we put $\mathcal{R}_{k+1}=\bigcup_{n=0}^{\infty}\mathcal{R}_{k+1}^{(n)}$ and $\mathcal{R}^{(n)}=\bigcup_{k=1}^{\infty}\mathcal{R}_{k+1}^{(n)}$. We have naturally $\mathcal{P}_k^{(n)}\subset\mathcal{P}_{k+1}^{(n)}$ and $\mathcal{R}_k^{(n)}\subset\mathcal{R}_{k+1}^{(n)}$ by adding trivial partitions and we put $\mathcal{P}=\bigcup_{k=1}^{\infty}\mathcal{P}_{k+1}$ and $\mathcal{R}=\bigcup_{k=1}^{\infty}\mathcal{R}_{k+1}$.

Definition 2.4. We define the index of parameters for a realizable $\mathbf{m}\in\mathcal{P}_{k+1}$ by

$$\begin{aligned}
\operatorname{Pidx}\mathbf{m}:=\min_{\lambda_{\mathbf{m}}}\dim\bigl\{A\in M(n,\mathbb{C})\,;\,[A,A_j]=0\qquad(j=1,\ldots,k,\\
(A_1,\ldots,A_k)\in V_{\mathbf{m},\lambda_{\mathbf{m}}})\bigr\}+\frac{k-1}{2}(\operatorname{ord}\mathbf{m})^2-\frac12\sum_{j=0}^{k}\sum_{\nu=1}^{n_j}m_{j,\nu}^2.
\end{aligned}\tag{2.6}$$

Definition 2.5. A tuple $\mathbf{m}\in\mathcal{P}$ with (2.3) is of *Okubo type* if there exists j satisfying

$$(m_{0,1}+\cdots+m_{k,1})-m_{j,1}=(k-1)\cdot\operatorname{ord}\mathbf{m}. \tag{2.7}$$

Definition 2.6. Suppose $\mathbf{m}\in\mathcal{R}_{k+1}$ and

$$m_{j,\nu}=m_{j,\nu}+m_{j,\nu}\quad(j=0,\ldots,k,\ \nu=1,\ldots,n_j) \tag{2.8}$$

with $\mathbf{m}',\mathbf{m}''\in\mathcal{R}$. Then we write $\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$, which we call a *rigid decomposition* of $\mathbf{m}$. In this case say $\mathbf{m}$ is said to be a rigid sum of $\mathbf{m}'$ and $\mathbf{m}''$.

This decomposition is said to be *irreducible* if $\mathbf{m}$, $\mathbf{m}'$ and $\mathbf{m}''$ are irreducible.

3. Conjectures for rigid decompositions

Conjecture 3.1 (Rigid Dec)**.** Any irreducible rigid tuple $\mathbf{m} \in \mathcal{R}$ with $\operatorname{ord}\mathbf{m} \geq 2$ has an irreducible rigid decomposition.

Conjecture 3.2 (Rigid Sum)**.** Let $\mathbf{m}$, $\mathbf{m}'$, $\mathbf{m}'' \in \mathcal{P}_{k+1}$ satisfying

$$m_{j,\nu} = m'_{j,\nu} + m''_{j,\nu} \qquad (j = 0, \dots, k,\ \forall\nu). \tag{3.1}$$

If $\mathbf{m}$, $\mathbf{m}' \in \mathcal{R}_{k+1}$ and

$$\sum_{j,\,\nu} m'_{j,\nu} m''_{j,\nu} = (k-1) \cdot \operatorname{ord} m' \cdot \operatorname{ord} m'' - 1, \tag{3.2}$$

and they are irreducible, then $\mathbf{m}'' \in \mathcal{R}_{k+1}$ and it is also irreducible.

Our main conjecture is as follows.

Conjecture 3.3 (Rigid 3.1.1)**.** Fix $\mathbf{m} \in \mathcal{R}_3$ with $m_{0,n_0} = m_{1,n_1} = 1$ and put

$$\mathcal{R}(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) := \{\mathbf{m}'\,;\, \mathbf{m} = \mathbf{m}' \oplus \mathbf{m}'',\ \ m'_{0,n_0} = m''_{1,n_1} = 1\}. \tag{3.3}$$

Then

$$\sum_{\mathbf{m}' \in \mathcal{R}(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1})} m'_{j,\nu} = (n_1 - 1) m_{j,\nu} - (1 - n_0 \delta_{\nu,n_0})\delta_{j,0} + (1 - n_1 \delta_{\nu,n_1})\delta_{j,1}. \tag{3.4}$$

i) (Conjecture: fundamental Rigid 3.1.1). Putting $j = 0$ and $\nu = 1$, we have

$$\#\mathcal{R}(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) = n_0 + n_1 - 2. \tag{3.5}$$

ii) Putting $j = 2$ and $\nu = 1$ and summing up for all ν, we have

$$\sum_{\mathbf{m}' \in \mathcal{R}(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1})} \operatorname{ord}\mathbf{m}' = (n_1 - 1) \cdot \operatorname{ord}\mathbf{m}' \tag{3.6}$$

Conjecture 3.4 (Rigid 3.1.0)**.** Any $\mathbf{m} \in \mathcal{R}_3$ with $m_{0,n_0} = m_{1,n_1} = 1$ has a rigid decomposition $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ with $m'_{0,n_0} = m'_{1,n_1} = 0$.

4. Program Okubo

4.1. **Get rigid tuples of partitions.** The program Okubo generates all rigid tuples of partitions as follows.

`Okubo <ord>  A`	: all rigid triplets of order `<ord>`
`Okubo <ord>  Ao`	: all rigid triplets up to order `<ord>`
`Okubo <ord>+ A`	: all rigid tuples of order `<ord>`
`Okubo <ord>+ Ao`	: all rigid tuples up to order `<ord>`
`Okubo <ord>+<n> Ao`	: same as above with # partitions $\geq$ `<n>`
`Okubo <ord>+<n> A`	: same as above of order `<ord>`
`Okubo <ord>+<n>-<m> A`	: same as above with `<n>` $\leq$ # partitions $\leq$ `<m>`

For example

```
>Okubo 4+ A
 4:211,211,211 o
 4:211,22,31,31 o
 4:1111,211,22 o
 4:1111,1111,31 o
 4:22,22,22,31
 4:31,31,31,31,31 o
```

```
Total number: 6 (Okubo 5)

>Okubo 6+5 A
 6:411,42,42,51,51 o
 6:321,33,51,51,51 o
 6:33,42,42,51,51 o
 6:51,51,51,51,51,51,51 o
 Total number: 28 (Okubo 4)
 Target: 4 exist
```

Here the sign `o` indicates Okubo type.

4.2. **Okubo type.** Tuples of partitions $32, 221, 11111$ and $311, 2111, 2111$ are of Okubo type, which are also expressed as

normal	O1-type	O2-type
$32, 221, 11111$	$2(2),\ 3(2,1),\ 1,1,1,1,1$	$(2)\,[3]\,;\,[2]\,(2)\,(1),\ 5(1)$
$32, 2111, 2111$	$2(2),\ 3(1,1,1),\ 2,1,1,1$	$(2)\,[3]\,;\,[2]\,3(1),\ (2)\,3(1)$

In O1-type or O2-type we arrange $\mathbf{m} \in \mathcal{P}_{k+1}$ so that

$$m_{1,1} + \cdots + m_{k-1,0} = \operatorname{ord} \mathbf{m}$$

with (2.3). In O1-type the numbers $\operatorname{ord} \mathbf{m} - m_{j,1}$ with the partitions $(m_{j,2}, \ldots . m_{j,n_j})$ of $\operatorname{ord} \mathbf{m} - m_{j,1}$ for $j = 1, \ldots, k$ and the partition $m_{0,1}, \ldots, m_{0,n_0}$ are written. In O2-type the partition are expressed by the numbers in ( ) with the multiplicities and the numbers in [ ] are $m_{j,1}$ $(j = 1, \ldots, k)$.

`Okubo <ord> a`	: all rigid triplets of order `<ord>` in O2-type
`Okubo <ord> ao`	: all rigid triplets up to order `<ord>` in O2-type
`Okubo <ord> ar`	: all rigid triplets of order `<ord>` in O1-type
`Okubo <ord> aro`	: all rigid triplets up to order `<ord>` in O1-type
`Okubo <ord> aR`	: all rigid triplets of order `<ord>`
`Okubo <ord> aRo`	: all rigid triplets up to order `<ord>`

4.3. **Rigidity index, Katz' reduction and Yokoyama's reduction.** Give reductions/constructions for rigid and non-rigid tuples.

`Okubo <tuple>`	: Katz'/Yokoyama's reduction and/or rigidity index
`Okubo <tuple> x`	: same as above but well-known families are abbreviated
`Okubo <tuple> ky`	: Katz' and Yokoyama's reductions and/or rigidity index
`Okubo <tuple> kY`	: same as above but Yokoyama's reductions with Katz'
`Okubo <tuple> kyx`	: same as above but well-known families are abbreviated
`Okubo <tuple> kyR`	: `R` indicates normal expression for Yokoyama's reduction
`Okubo <tuple> yk`	: Katz-Yokoyama reduction for Okubo system
`Okubo <tuple> Yk`	: same as above after changing into Okubo system
`Okubo <tuple> K`	: all the first steps of Katz' reduction
`Okubo <tuple> K+`	: same as above but the order decreases more than 1
`Okubo <tuple> KS`	: same as `K` but suppress the same kind of reductions
`Okubo <tuple> K+S`	: same as `K+` but suppress the same kind of reductions

Here the well-known families are hypergeometric/even/odd/extra case/Pochhammer families. Note that Yokoyama's reduction is possible for a tuple of Okubo type.

If `Y` is indicated in place of `y`, Katz' reduction may be used to change the tuple into Okubo type.

`R` can be generally indicated for Yokoyama's reduction

`<tuple>` can be replaced by `<ord>` or `<ord>+` or `<ord>+<n>-<m>` etc. and in this case the option parameter `d` and `a` or `A` are indicated and the option parameter `o` is also possible.

The number larger than 10 is expressed by `a`, `b`,...,`z`,`A`,...,`Z` and then `a` means 10 and `A` means etc.

1	⋯	9	10	11	⋯	35	36	⋯	60
1	⋯	9	a	b	⋯	z	A	⋯	Z

```
>Okubo 221,221,221
 1: 221,221,221
 2: 211,211,211
 3: 11,11,11

>Okubo 311,221,11111
Rigidity index: 0 (1)
 1: 311,221,11111
 2: 211,211,1111
 3: 111,111,111
Generically exists

>Okubo 51,33,33,3111
 1: 51,33,33,3111
 2: 31,31,31,1111
Not irreducible/doesn't exist
Rigidity index: 2

>Okubo 32,2111,2111 x
 1: 32,2111,2111
 2: 21,111,111
generalized hypergeometric 3F2

>Okubo 3(2,1) 3(1,1,1) 2,2,2 x
 1:E0->3(2,1),3(1,1,1),2(2),4,4
 2:R0->3(1,1,1),2(2),2,1,1,1
 3:E0->3(1,1,1),2(2),2(1,1),4,3
 4:R1->2(2),2(1,1),1,1,1,1
even family of order 4

>Okubo 3(2,1) 3(1,1,1) 2,2,2 xR
       321,3111,222
 1:E0->521,5111,62,44
 2:R0->2111,32,2111
 3:E0->4111,52,511,43
 4:R1->22,211,1111
even family of order 4
```

```
>Okubo 221,221,221 xRY
 1: 221,221,221
 2: 211,211,211
 3: 11,11,11
Gauss hypergeometric of order 2

 0:    321,321,2211
 1:E0->521,521,611,44
 2:R0->221,311,2111
 3:E0->421,511,511,43
 4:R1->211,211,211
 5:E0->311,311,41,32
 6:R1->111,21,111
hypergeometric family 3F2

>Okubo "42,2^3,1^6"
 1: 42,222,111111
 2: 32,221,11111
 3: 22,211,1111
 4: 21,111,111
 5: 11,11,11

>Okubo 42,222,111111 ky
 1: 42,222,111111
 2: 32,221,11111
 3: 22,211,1111
 4: 21,111,111
 5: 11,11,11

 0:    4(2,2),2(2),1,1,1,1,1,1
 1:E0->4(2,2),2(2),4(1,1,1,1),5,5
 2:R0->2(2),4(1,1,1,1),2,2,1,1
 3:E0->2(2),4(1,1,1,1),2(1,1),4,4
 4:R2->2(2),2(1,1),1,1,1,1
 5:E0->2(2),2(1,1),2(1,1),3,3
 6:R0->2(1,1),2(1,1),2,1,1
 7:E0->2(1,1),2(1,1),1(1),3,2
 8:R1->2(1,1),1(1),1,1,1
 9:E0->2(1,1),1(1),1(1),2,2
10:R2->1(1),1(1),1,1

>Okubo 221,221,221 Yk
 1: 221,221,221
 2: 211,211,211
 3: 11,11,11

 0:2211,321,321
 1:2111,311,221
```

```
 2:111,111,21
 3:11,11,11

>Okubo 41,41,41,32,32 K
 1: 41,41,41,32,32
 2: 11,11,11

 41,41,41,32,32
=31,31,31,31,31+10,10,10,01,01
=21,21,21,12,30+20,20,20,20,02
=21,21,21,30,12+20,20,20,02,20
=11,11,11,02,02+30,30,30,30,30

>Okubo 8 Axk
...
8:221111,3311,53
 1: 221111,3311,53
 2: 21111,3111,33
 3: 1111,1111,31
generalized hypergeometric 4F3
...

>Okubo 8 axy
...
(3)  [5] ; [3]  (3) (2),  (2) 6(1)
 1:E0->5(3,2),3(3),5(1,1,1,1,1),7,6
 2:R0->3(3),5(1,1,1,1,1),3,2,2,1
 3:E0->3(3),5(1,1,1,1,1),3(2,1),6,5
 4:R1->3(3),3(2,1),1,1,1,1,1,1
even family of order 6
...
```

4.4. **Rigid decompositions.** The option parameter `V+/W` indicates to give all the (irreducible) rigid decomposition.

`Okubo <tuple> V+`	: Katz' reduction and irreducible rigid decompositions
`Okubo <tuple> V+y`	: same as above with Yokoyama's reduction
`Okubo <ord>  AV+`	: irred. rigid decompositions of rigid triplets of order `<ord>`
`Okubo <ord>  AoV+`	: same as above up to order `<order>`
`Okubo <ord>+ AV+`	: irred. rigid decompositions of rigid tuples of order `<ord>`
`Okubo <ord>+ AoV+`	: same as above up to order `<order>`
`Okubo <tuple> W`	: Katz' reduction and rigid decompositions
`Okubo <tuple> Wy`	: same as above with Yokoyama's reduction
`Okubo <ord>  AW`	: rigid decompositions of rigid triplets of order `<ord>`
`Okubo <ord>  AoW`	: same as above up to order `<order>`
`Okubo <ord>+ AW`	: rigid decompositions of rigid tuples of order `<ord>`
`Okubo <ord>+ AoW`	: same as above up to order `<order>`

In the above the option parameter `S` and/or `X` may be also indicated. The option `S` indicates to suppress the same kind of decompositions. The option `X` indicates the application of Katz' maximal reduction to the decomposition.

If the given tuple is not rigid, the decompositions shown under the option parameter `W` are those with realizable tuples whose indices of parameters are also correspondingly decomposed.

```
>Okubo "22,211,1^4" V+
 1: 22,211,1111
 2: 21,111,111
 3: 11,11,11

 22,211,1111
=10,100,1000+12,111,0111 [2+0]
=10,100,0100+12,111,1011 [1+0]
=10,100,0010+12,111,1101 [1+0]
=10,100,0001+12,111,1110 [1+0]
=01,100,1000+21,111,0111 [1+0]
=01,100,0100+21,111,1011 (0+1)
=01,100,0010+21,111,1101 (0+1)
=01,100,0001+21,111,1110 (0+1)
=11,110,1100+11,101,0011 (1+0)
=11,110,1010+11,101,0101 (1+0)
=11,110,1001+11,101,0110 (1+0)
=11,110,0110+11,101,1001 (0+1)
=11,110,0101+11,101,1010 (0+1)
=11,110,0011+11,101,1100 (0+1)

>Okubo 2111,221,311 WS
 1: 2111,221,311
 2: 111,21,111
 3: 11,11,11

 2111,221,311
=1000,001,100+1111,220,211
=0100,100,100+2011,121,211
=2000,200,200+0111,021,111
=1100,110,110+1011,111,201

>Okubo 411,2211,2211 xyWRS
 1: 411,2211,2211
 2: 211,211,211
 3: 11,11,11
Gauss hypergeometric of order 2

 0:    411,2211,2211
 1:E0->611,4211,611,44
 2:R2->211,211,211
 3:E0->311,311,41,32
```

```
 4:R1->111,21,111
hypergeometric family 3F2

 411,2211,2211
=100,1000,0010+311,1211,2201
=200,2000,2000+211,0211,0211
=110,1100,1100+301,1111,1111
=210,1110,1110+201,1101,1101

>Okubo 42,222,222 WS
Rigidity index: 8 (0)
 1: 42,222,222
 2: 22,22,22

 42,222,222
=20,200,200+22,022,022
=21,111,111+21,111,111
```

The last example `42,222,222` above is clearly reducible and rigid. The half of the number of accessory parameters which equals $\mathrm{Pidx}\,\mathbf{m}$ given in Definition 2.4 is indicated in `( )`.

4.5. **Connection formula.** Get rigid decompositions to give the connection formula for triplets with two partitions containing 1.

`Okubo <triplet> V` : required rigid decompositions of `triplet`
`Okubo <ord> AV` : the decompositions for the triplets of order `<ord>`
`Okubo <ord> AokV` : same as above up to order `<ord>` with Katz' reduction
Option `X` may be also indicated, which indicates Katz' maximal reduction to the connection formula.

```
>Okubo 211,1111,22 Vx
 1: 211,1111,22
even family of order 4

 211,1111,22
=101,1100,11+110,0011,11
=101,1010,11+110,0101,11
=101,0110,11+110,1001,11
=111,1110,12+100,0001,10
=111,1110,21+100,0001,01
```

4.6. **Other functions.** Okubo has several functions.
`Okubo <ord> AoV-?` : check Conjecture Rigid 3.1.1 up to order `<ord>`
`Okubo <ord> AoV-1?` : check Conjecture Rigid 3.1.0 up to order `<ord>`

`Okubo <ord> AV+-` : give the number of irreducible rigid decompositions and the minimal order of the component
`Okubo <ord> AV+-S`: same as above with identifying symmetries
`Okubo <ord> AW-` : give the number of rigid decompositions and the minimal order of the component and check Conjecture Rigid Sum
`Okubo <ord> AW-S` : same as above with identifying symmetries

In the above <ord> may be replaced by <ord>+ etc. and A by Ao

Okubo <ord> AV+-? : give triplets without an irreducible rigid decomposition having a component of order 1 and check Conjecture Rigid Dec

Okubo <ord>+ AV+-? : same as the above for tuples

Okubo <ord>+ AV+-S?: check Conjecture Rigid Dec up to <ord>

Okubo <ord>+ AoV+-T1? : give tuples without any irreducible decomposition whose two components have 0

Okubo <ord>+ AW-T3? : give rigid decompositions of tuples which are not of Katz reduction and not irreducible.

Okubo <ord>+ AW-T4? : give rigid decompositions of tuples which are not generalized middle convolution

Okubo <ord>+ AW-T5? : give rigid decompositions whose components are reducible with no tivial irreducible components

Okubo <ord>+ AV-T3? : give rigid decompositions for generalized middle convolutions

Okubo <ord>+ AoV-ST4? : check generalized Conjecture Sum

Okubo <idx> B+ : give basic tuples with a given rigidity index

Okubo <tuple> E<ord>+o: give Katz' extensions up to order <ord>

Okubo <tuple> e<ord>+o: give Katz' (ext/red)s up to order <ord>

Okubo <tuple> z : Get the positions of accessory parameters

Okubo <ord> a-[<level>][rR] : non-existent plausible Okubo triplets

4.6.1. Okubo <ord> AoV-? : check Conjecture Rigid 3.1.1 up to order <ord>

Okubo <ord> AoV-1? : check Conjecture Rigid 3.1.0 up to order <ord>

```
>Okubo 20 AoV-?
Get 1 rigid systems of order 2 (1)
Get 1 rigid systems of order 3 (3)
Get 3 rigid systems of order 4 (7)
...
Get 6954 rigid systems of order 19 (15316)
Get 10517 rigid systems of order 20 (21681)
Total number: 32331 (Okubo 21136)
Checked Conjecture Rigid 3.1.1 : 21681
```

4.6.2. Okubo <ord> AV+- : give the number of irreducible rigid decompositions and the minimal order of the component

Okubo <ord> AW- : give the number of rigid decompositions and the minimal order of the component

The option parameter S may be also indicated.

```
>Okubo 5 AV+-
5:2111,221,311  13(1)
5:2111,2111,32  16(1)
5:221,221,221   14(1)
5:11111,221,32  20(1)
5:11111,11111,41        25(1)
Total number: 5 (Okubo 4)
```

4.6.3. `Okubo <ord>  AV+-?` : give triplets without an irreducible rigid decomposition having a component of order 1 and check Conjecture Rigid Dec

`Okubo <ord>+ AV+-?` : same as the above for tuples

`Okubo <ord>+ AV+-S?`: check Conjecture Rigid Dec up to `<ord>`

```
>Okubo 10 AoV+-?
Get 1 rigid systems of order 2
Get 1 rigid systems of order 3
Get 3 rigid systems of order 4
Get 5 rigid systems of order 5
Get 13 rigid systems of order 6
Get 20 rigid systems of order 7
2222,3311,521   (2)
Get 45 rigid systems of order 8
3222,441,51111  (2)
333,4311,51111  (2)
333,42111,441   (2)
Get 74 rigid systems of order 9
3322,4411,6211  (2)
433,4411,5221   (2)
4411,4411,4411  (2)
3331,3331,631   (2)
3331,442,532    (2)
22222,3331,721  (3)
3331,421111,55  (2)
22222,442,631   (2)
22222,52111,55  (2)
Get 142 rigid systems of order 10
Total number: 304 (Okubo 231)
Target: 13 exist
```

4.6.4. `Okubo <ord>+ AoV+-T1?` : give tuples without any irreducible decomposition whose two components have 0

```
>Okubo 8 AoV+-T1?
Get 1 rigid systems of order 2
Get 1 rigid systems of order 3
Get 3 rigid systems of order 4
Get 5 rigid systems of order 5
Get 13 rigid systems of order 6
322,322,322
Get 20 rigid systems of order 7
2222,422,431
332,332,332
2222,2222,53
Get 45 rigid systems of order 8
Total number: 88 (Okubo 70)
```

4.6.5. `Okubo <ord>+ AW-T3?` : give rigid decompositions of tuples which are not of Katz reduction and not irreducible.

`Okubo <ord>+ AW-T4?` : give rigid decompositions of tuples which are not generalized middle convolutions

`Okubo <ord>+ AW-T5?` : give rigid decompositions whose components are reducible with no tivial irreducible components

```
>Okubo 5+ AW-ST3?
...
5:221,32,32,41=001,10,10,01+220,22,22,40
5:32,32,32,32=02,02,02,02+30,30,30,30
5:221,221,221=001,001,001+220,220,220
...

>Okubo 5+ AW-ST4?
...
5:32,32,32,32=02,02,02,02+30,30,30,30
...

>Okubo 10 AW-ST5?
...
10:3322,532,532=0022,202,202+3300,330,330
10:3322,3322,55=0022,0022,22+3300,3300,33
...
```

4.6.6. `Okubo <idx> B` : give basic triplets with a given rigidity index

`Okubo <idx> B+` : give basic tuples with a given rigidity index

`Okubo <idx> B+z` : give basic tuples with a given rigidity index and the accessory parameters

`Okubo <idx>+<n>-<m> B` : same as above with `n` $\leq$ #partitions $\leq$ `m`

A tuple is called basic if its order cannot be decreased by any Katz' reduction

`W` option gives rigid decomposition

```
>Okubo 2 B
1:1,1,1

>Okubo 0 B
2:11,11,11,11
3:111,111,111
4:1111,1111,22
6:111111,222,33
#4 exist (max order 6)

>Okubo -2 B
2:11,11,11,11,11
4:1111,1111,211
4:211,22,22,22
6:111111,2211,33
6:2211,222,222
8:22211,2222,44
12:2222211,444,66
4:1111,22,22,31
```

```
10:22222,3331,55
5:11111,11111,32
8:11111111,332,44
3:111,111,21,21
5:11111,221,221
#13 exist (max order 12)

>Okubo -4 B+W
 ...
 3333321,666,99
=1111110,222,33+2222211,444,66
=1111101,222,33+2222220,444,66
 22222,3322,55
 111111,111111,42
 ...
#36 exist (max order 18)

>Okubo -2 B+z
2:11,11,11,11,11
145
002 (Pidx:2)

4:211,1111,1111
12334
00011 (Pidx:2)

4:211,22,22,22
13478
00101 (Pidx:2)
 ...

>Okubo -20 B+z
...
14:662,44411,44411
123344677aabdee
000112112011001 (Pidx:11)
 ...
#720 exist (max order 66)
```

4.6.7. `Okubo <tuple> E<ord>+o` : give Katz' extensions up to order `<ord>`
`Okubo <tuple> E<ord>+` : give Katz' extensions of order `<ord>`
`Okubo <tuple> e<ord>+o` : give Katz' (ext/red)s up to order `<ord>`
`Okubo <tuple> e<ord>+` : give Katz' (ext/red)s of order `<ord>`
`Okubo <idx> E<ord>+o` : give irreducible tuples with index `<idx>`
`Okubo <idx> E<ord>+` : give irreducible triplets with index `<idx>`

If `+` is omoitted, only triplets are given.

```
>Okubo 11,11,11,11 E4+o
2:11,11,11,11
3:111,21,21,21
```

```
4:1111,22,31,31
4:211,211,31,31
4:211,22,22,31
4:22,31,31,31,31

>Okubo 111,111,111 E6o
3:111,111,111
4:1111,211,211
5:11111,221,311
5:2111,2111,311
5:2111,221,221
6:111111,222,411
6:21111,2211,411
6:21111,321,321
6:2211,222,321
6:2211,3111,321
6:222,222,3111
6:3111,3111,3111

>Okubo 111,111,111 E10+o!?
Get 0 rigid systems of order 2 (other 1)
Get 1 rigid systems of order 3 (other 0)
Get 1 rigid systems of order 4 (other 8)
Get 3 rigid systems of order 5 (other 21)
Get 8 rigid systems of order 6 (other 54)
Get 15 rigid systems of order 7 (other 106)
Get 31 rigid systems of order 8 (other 152)
Get 65 rigid systems of order 9 (other 112)
Get 113 rigid systems of order 10 (other 0)

>Okubo 2 E20+o > rigid20.txt
```

4.6.8. `Okubo <ord> a-[<level>][rR]` : non-existent plausible Okubo triplets

```
>Okubo 15 a-4oR
 15:843,7611,444111
Total number: 1

>Okubo 843,7611,444111
 1: 843,7611,444111
 2: 443,6311,44111
 3: 431,3311,41111
 4: 311,311,11111
Not irreducible/doesn't exist
Rigidity index: 2
```

4.6.9. `Okubo <tuple> z` : get the position of accessory parameters

```
>Okubo 211,1111,1111 z
Rigidity index: -2 (2)
 1: 211,1111,1111
```

```
Generically exists
12334
00011 (Pidx:2)
```

5. Some results

5.1. Tables. We give some tables obtained by `Okubo`.

5.1.1. *Rigid tuples.* ($R_k^{(n)}$: irreducible tuples of k partitions with order n)

$\#\mathcal{R}_3^{(n)}$ and $\#\mathcal{R}^{(n)}$

ord	$\mathcal{R}_3$	$\mathcal{R}$	ord	$\mathcal{R}_3$	$\mathcal{R}$	ord	$\mathcal{R}_3$	$\mathcal{R}$	ord	$\mathcal{R}_3$	$\mathcal{R}$
1	1	1	11	212	441	21	14040	24748	31	235543	371773
2	1	1	12	421	857	22	20210	36078	32	309156	493620
3	1	2	13	588	1177	23	26432	45391	33	378063	588359
4	3	6	14	1004	2032	24	37815	65814	34	487081	763126
5	5	11	15	1481	2841	25	48103	80690	35	591733	903597
6	13	28	16	2388	4644	26	66409	112636	36	756752	1170966
7	20	44	17	3276	6128	27	84644	139350	37	907150	1365027
8	45	96	18	5186	9790	28	114600	190465	38	1143180	1734857
9	74	157	19	6954	12595	29	143075	230110	39	1365511	2031018
10	142	306	20	10517	19269	30	190766	310804	40	1704287	2554015

$\mathcal{R}^{(n)}$ $(2 \le n \le 8)$

```
2:11,11,11              3:111,111,21            3:21,21,21,21
4:1111,1111,31          4:1111,211,22           4:211,211,211
4:211,22,31,31          4:22,22,22,31           4:31,31,31,31,31
5:11111,11111,41        5:11111,221,32          5:2111,2111,32
5:2111,221,311          5:221,221,221           5:221,221,41,41
5:221,32,32,41          5:311,311,32,41         5:32,32,32,32
5:32,32,41,41,41        5:41,41,41,41,41,41     6:111111,111111,51
6:111111,222,42         6:111111,321,33         6:21111,2211,42
6:21111,222,33          6:21111,222,411         6:21111,3111,33
6:2211,2211,33          6:2211,2211,411         6:2211,222,51,51
6:2211,321,321          6:2211,33,42,51         6:222,222,321
6:222,3111,321          6:222,33,33,51          6:222,33,411,51
6:3111,3111,321         6:3111,33,411,51        6:321,321,42,51
6:321,33,51,51,51       6:321,42,42,42          6:33,33,33,42
6:33,33,411,42          6:33,411,411,42         6:33,42,42,51,51
6:411,411,411,42        6:411,42,42,51,51       6:51,51,51,51,51,51,51
7:1111111,1111111,61    7:1111111,331,43        7:211111,2221,52
7:211111,322,43         7:22111,22111,52        7:22111,2221,511
7:22111,3211,43         7:22111,331,421         7:2221,2221,43
7:2221,2221,61,61       7:2221,31111,43         7:2221,322,421
7:2221,331,331          7:2221,331,4111         7:2221,43,43,61
7:31111,31111,43        7:31111,322,421         7:31111,331,4111
7:3211,3211,421         7:3211,322,331          7:3211,322,4111
7:3211,331,52,61        7:322,322,322           7:322,322,52,61
7:322,331,511,61        7:322,421,43,61         7:322,43,52,52
7:331,331,43,61         7:331,331,61,61,61      7:331,43,511,52
7:4111,4111,43,61       7:4111,43,511,52        7:421,421,421,61
7:421,421,52,52         7:421,43,43,52          7:421,43,511,511
```

7:421,43,52,61,61	7:43,43,43,43	7:43,43,43,61,61
7:43,43,61,61,61,61	7:43,52,52,52,61	7:511,511,52,52,61
7:52,52,52,61,61,61	7:61,61,61,61,61,61,61,61	8:11111111,11111111,71
8:11111111,431,44	8:2111111,2222,62	8:2111111,332,53
8:2111111,422,44	8:221111,22211,62	8:221111,2222,611
8:221111,3311,53	8:221111,332,44	8:221111,4211,44
8:22211,22211,611	8:22211,2222,71,71	8:22211,3221,53
8:22211,3311,44	8:22211,332,521	8:22211,41111,44
8:22211,431,431	8:22211,44,53,71	8:2222,2222,53
8:2222,32111,53	8:2222,3221,44	8:2222,3311,521
8:2222,332,5111	8:2222,422,431	8:2222,44,44,71
8:311111,3221,53	8:311111,332,521	8:311111,41111,44
8:32111,32111,53	8:32111,3221,44	8:32111,3311,521
8:32111,332,5111	8:32111,422,431	8:3221,3221,521
8:3221,3311,5111	8:3221,332,431	8:3221,332,62,71
8:3221,4211,431	8:3221,422,422	8:3221,44,521,71
8:3221,44,62,62	8:3311,3311,431	8:3311,3311,62,71
8:3311,332,422	8:3311,332,611,71	8:3311,4211,422
8:3311,431,53,71	8:3311,44,611,62	8:332,332,332
8:332,332,4211	8:332,332,71,71,71	8:332,41111,422
8:332,4211,4211	8:332,422,53,71	8:332,431,44,71
8:332,44,611,611	8:332,44,62,71,71	8:332,53,53,62
8:41111,41111,431	8:41111,4211,422	8:41111,44,5111,71
8:41111,44,611,62	8:4211,4211,4211	8:4211,422,53,71
8:4211,44,611,611	8:4211,44,62,71,71	8:4211,53,53,62
8:422,422,44,71	8:422,431,521,71	8:422,431,62,62
8:422,44,53,62	8:422,44,611,71,71	8:422,53,53,611
8:431,431,611,62	8:431,44,44,62	8:431,44,53,611
8:431,44,71,71,71,71	8:431,521,53,62	8:431,53,53,71,71
8:44,44,44,53	8:44,44,62,62,71	8:44,5111,521,62
8:44,521,521,611	8:44,521,53,53	8:44,53,611,62,71
8:5111,5111,53,62	8:5111,521,53,611	8:521,521,521,62
8:521,521,53,71,71	8:521,53,62,62,71	8:53,53,611,611,71
8:53,53,62,71,71,71	8:53,62,62,62,62	8:611,611,611,62,62
8:611,62,62,62,71,71	8:71,71,71,71,71,71,71,71,71	

5.1.2. *Basic tuples.* (irreducible tuples cannot be reduced to lower order ones)

index	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20
# of base	4	13	36	67	90	162	243	305	420	565	720
# triplets	3	9	24	44	56	97	144	163	223	291	342
# tuples of 4	1	3	9	17	24	45	68	95	128	169	239
max order	6	12	18	24	30	36	42	48	54	60	66

rigidity index $= 0$

2:11,11,11,11	3:111,111,111	4:1111,1111,22	6:111111,222,33

ord	11,11,11,11	111,111,111	22,1111,1111	33,222,111111	total
2	1				1
3	1	1			2
4	4	1	1		6
5	6	3	1		10
6	21	8	5	1	35
7	28	15	6	1	50
8	74	31	21	4	130
9	107	65	26	5	203
10	223	113	69	12	417
11	315	204	90	14	623
12	616	361	205	37	1219
13	808	588	256	36	1688
14	1432	948	517	80	2977
15	1951	1508	659	100	4218
16	3148	2324	1214	179	6865
17	4064	3482	1531	194	9271
18	6425	5205	2641	389	14660
19	8067	7503	3246	395	19211
20	12233	10794	5400	715	29142

rigidity index $= -2$

```
  2:11,11,11,11,11  3:111,111,21,21  *4:211,22,22,22       4:1111,22,22,31
  4:1111,1111,211   5:11111,11111,32  5:11111,221,221      6:111111,2211,33
 *6:2211,222,222   *8:22211,2222,44   8:11111111,332,44   10:22222,3331,55
*12:2222211,444,66
```

rigidity index $= -4$

```
  2:11,11,11,11,11,11        3:111,21,21,21,21          4:22,22,22,31,31
  3:111,111,111,21           4:1111,22,22,22            4:1111,1111,31,31
  4:211,211,22,22            4:1111,211,22,31          *6:321,33,33,33
  6:222,222,33,51            4:1111,1111,1111           5:11111,11111,311
  5:11111,2111,221           6:111111,222,321           6:111111,21111,33
  6:21111,222,222            6:111111,111111,42         6:222,33,33,42
  6:111111,33,33,51          6:2211,2211,222            7:1111111,2221,43
  7:1111111,331,331          7:2221,2221,331            8:11111111,3311,44
  8:221111,2222,44           8:22211,22211,44          *9:3321,333,333
  9:111111111,333,54         9:22221,333,441           10:1111111111,442,55
 10:22222,3322,55           10:222211,3331,55          12:22221111,444,66
*12:33321,3333,66           14:2222222,554,77         *18:3333321,666,99
```

rigidity index $= -6$

2:11,11,11,11,11,11,11	3:21,21,21,21,21,21	3:111,111,21,21,21
4:22,22,22,22,31	4:211,22,22,31,31	4:1111,22,31,31,31
3:111,111,111,111	4:1111,1111,22,31	4:1111,211,22,22
4:211,211,211,22	4:1111,211,211,31	5:11111,11111,41,41
5:11111,221,32,41	5:221,221,221,41	5:11111,32,32,32
5:221,221,32,32	6:3111,33,33,33	6:2211,2211,2211
6:222,33,33,33	6:222,33,33,411	6:2211,222,33,51
*8:431,44,44,44	8:11111111,44,44,71	5:11111,11111,221
5:11111,2111,2111	6:111111,111111,33	6:111111,222,222
6:111111,111111,411	6:111111,222,3111	6:21111,2211,222
6:111111,2211,321	6:2211,33,33,42	7:1111111,1111111,52
7:1111111,322,331	7:2221,2221,322	7:1111111,22111,43
7:22111,2221,331	8:11111111,3221,44	8:11111111,2222,53
8:2222,2222,431	8:2111111,2222,44	8:221111,22211,44
9:33111,333,333	9:3222,333,333	9:22221,22221,54
9:222111,333,441	9:111111111,441,441	10:22222,33211,55
10:1111111111,433,55	10:1111111111,4411,55	10:2221111,3331,55
10:222211,3322,55	12:222111111,444,66	12:333111,3333,66
12:33222,3333,66	12:222222,4431,66	*12:4431,444,444
12:111111111111,552,66	12:3333,444,552	14:33332,4442,77
14:22222211,554,77	15:33333,555,771	*16:44431,4444,88
16:333331,5551,88	18:33333111,666,99	18:3333222,666,99
*24:4444431,888,cc		

5.2. **Checking Conjectures.** Here are results on checking Conjectures[1] by using `Okubo`.

Conjecture 3.1.1 : checked up to order ≤ 40 (4111704 different pairs) together with Conjecture Sum for the corresponding decompositions

Conjecture 3.1.0 : checked up to order ≤ 35 (1405870 cases)

Conjecture Dec : checked up to order ≤ 35 (4416987 cases)

5.3. **Remarks.** The algorithm used by `Okubo` to obtain rigid decompositions under the option `W` is different from the one under the option `V+`.

The former generates all the decompositions of the given tuple and checks the realizability and indices of parameters of the summands. The latter checks if a tuple chosen from the list of rigid tuples generated by the program can be a summand of a rigid decomposition of a given tuple.

The two different algorithms may offer more reliability of the program to obtain required decompositions.

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[1]These conjectures are now proved (May 2008).

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